JEE(ADVANCED)–2024 (EXAMINATION)

(Held On Sunday 26th MAY, 2024)

MATHEMATICS | **TEST PAPER WITH ANSWER AND SOLUTION**

PAPER-2

SECTION-1 : (Maximum Marks : 12)

- This section contains **FOUR (04)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 If **ONLY** the correct option is chosen;

Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);

Negative Marks : −1 In all other cases.

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1. Considering only the principal values of the inverse trigonometric functions, the value of

$$\tan\left(\sin^{-1}\left(\frac{3}{5}\right)-2\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$$

is

(A) $\frac{7}{24}$ (B) $\frac{-7}{24}$

(C) $\frac{-5}{24}$ (D) $\frac{5}{24}$

Ans. (B)

Sol. $\tan\left(\tan^{-1}\left(\frac{3}{4}\right)-2\tan^{-1}\left(\frac{1}{2}\right)\right)$

$$\tan\left(\tan^{-1}\left(\frac{3}{4}\right)-\tan^{-1}\left(\frac{2\times\frac{1}{2}}{1-\frac{1}{4}}\right)\right)$$

$$\tan\left(\tan^{-1}\left(\frac{3}{4}\right)-\tan^{-1}\left(\frac{4}{3}\right)\right)$$

$$\tan\left(\tan^{-1}\left(\frac{\frac{3}{4}-\frac{4}{3}}{1+1}\right)\right)$$

$$\frac{9-16}{24}=\frac{-7}{24}$$

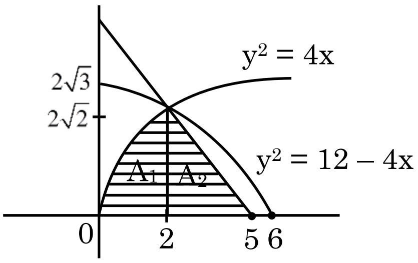
2. Let $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \geq 0, y \geq 0, y^2 \leq 4x, y^2 \leq 12 - 2x \text{ and } 3y + \sqrt{8}\, x \leq 5\sqrt{8}\}$. If the area of the region S is $\alpha\sqrt{2}$, then α is equal to

(A) $\frac{17}{2}$ (B) $\frac{17}{3}$

(C) $\frac{17}{4}$ (D) $\frac{17}{5}$

Ans. **(B)**

Sol.



Point of intersection of all curves is $\left(2, 2\sqrt{2}\right)$

Area = $A_1 + A_2$

$$\alpha\sqrt{2} = \int_0^2 2\sqrt{x}\, dx + \frac{1}{2} \times 3 \times 2\sqrt{2}$$

$$\alpha\sqrt{2} = 2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2 + 3\sqrt{2}$$

$$\alpha\sqrt{2} = \frac{17\sqrt{2}}{3}$$

$$\alpha = \frac{17}{3}$$

3. Let $k \in \mathbb{R}$. If $\lim\limits_{x \to 0^+} (\sin(\sin kx) + \cos x + x)^{\frac{2}{x}} = e^6$, then the value of k is

(A) 1 (B) 2 (C) 3 (D) 4

Ans. **(B)**

Sol. $\lim\limits_{x \to 0} \frac{2}{x}\left(\sin\left(\sin kx\right) + \cos x + x - 1\right) = 6$

$$\lim_{x \to 0} \frac{\sin\left(\sin kx\right) . \sin kx}{\left(\sin kx\right) kx} .k + 1 - \lim_{x \to 0} \frac{1 - \cos x}{x^2} .x = 3$$

$k + 1 = 3 \Rightarrow \boxed{k = 2}$

4. Let $f: \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{\pi}{x^2}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then which of the following statements is TRUE ?

(A) $f(x) = 0$ has infinitely many solutions in the interval $\left[\frac{1}{10^{10}}, \infty\right)$.

(B) $f(x) = 0$ has no solutions in the interval $\left[\frac{1}{\pi}, \infty\right)$.

(C) The set of solutions of $f(x) = 0$ in the interval $\left(0, \frac{1}{10^{10}}\right)$ is finite.

(D) $f(x) = 0$ has more than 25 solutions in the interval $\left(\frac{1}{\pi^2}, \frac{1}{\pi}\right)$.

Ans. **(D)**

Sol. **Option-A :** $f(x) = x^2 \sin\frac{\pi}{x^2}$

$f(x) = 0 \Rightarrow \sin\frac{\pi}{x^2} = 0 \Rightarrow \frac{\pi}{x^2} = n\pi,\ n \in N$

$x^2 = \frac{1}{n} \Rightarrow x = \frac{1}{\sqrt{n}}$

$\frac{1}{\sqrt{n}} \geq \frac{1}{10^{10}} \Rightarrow 10^{10} \geq \sqrt{n} \Rightarrow n \leq 10^{20}$, finite number of solutions

Option-B : $x = \frac{1}{\sqrt{n}} \Rightarrow \frac{1}{\sqrt{n}} > \frac{1}{\pi} \Rightarrow \pi > \sqrt{n} \Rightarrow n < \pi^2$, Number of solutions is 9

Option-C : $x = \frac{1}{\sqrt{n}},\ \frac{1}{\sqrt{n}} < \frac{1}{10^{10}} \Rightarrow \sqrt{n} > 10^{10} \Rightarrow n > 10^{20}$, Infinite number of solutions

Option-D : $\frac{1}{\pi^2} < \frac{1}{\sqrt{n}} < \frac{1}{\pi} \Rightarrow \sqrt{n} \in (\pi, \pi^2) \Rightarrow n \in (\pi^2, \pi^4)$, Definitely more than 25 solutions

SECTION-2 : (Maximum Marks : 12)

- This section contains **THREE (03)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	: +4	**ONLY** if (all) the correct option(s) is(are) chosen;
Partial Marks	: +3	If all the four options are correct but **ONLY** three options are chosen;
Partial Marks	: +2	If three or more options are correct but **ONLY** two options are chosen, both of which are correct;
Partial Marks	: +1	If two or more options are correct but **ONLY** one option is chosen and it is a correct option;
Zero Marks	: 0	If unanswered;
Negative Marks	: −2	In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then
  choosing ONLY (A), (B) and (D) will get +4 marks;
  choosing ONLY (A) and (B) will get +2 marks;
  choosing ONLY (A) and (D) will get +2 marks;
  choosing ONLY (B) and (D) will get +2 marks;
  choosing ONLY (A) will get +1 marks;
  choosing ONLY (B) will get +1 marks;
  choosing ONLY (D) will get +1 marks;
  choosing no option (i.e. the question is unanswered) will get 0 marks and
  choosing any other option(s) will get –2 marks.

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5. Let S be the set of all $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ such that

$$\lim_{x\to\infty} \frac{\sin(x^2)(\log_e x)^{\alpha} \sin\left(\frac{1}{x^2}\right)}{x^{\alpha\beta}(\log_e(1+x))^{\beta}} = 0 .$$

Then which of the following is (are) correct ?

(A) $(-1, 3) \in S$ (B) $(-1, 1) \in S$

(C) $(1, -1) \in S$ (D) $(1, -2) \in S$

Ans. **(B,C)**

Sol. $$\lim_{x\to\infty} \frac{\sin x^2.(\log_e x)^{\alpha}.\sin\frac{1}{x^2}}{x^{\alpha\beta}.(\log_e(1+x))^{\beta}} = 0$$

$$\lim_{x\to\infty} \frac{(\log_e x)^{\alpha}}{(\log_e(x+1))^{\beta}.x^{\alpha\beta+2}} = 0$$

$$\lim_{x\to\infty} \left(\frac{\log_e x}{\log_e(x+1)}\right)^{\beta}.\frac{(\log_e x)^{\alpha-\beta}}{x^{\alpha\beta+2}} = 0$$

$\lim_{x\to\infty}\frac{(\log_e x)^{\alpha-\beta}}{x^{\alpha\beta+2}}=0$ Put $\log_e x = t$

$\lim_{t\to\infty}\frac{t^{\alpha-\beta}}{(e^t)^{\alpha\beta+2}}=0$

As we know $\lim_{x\to\infty}\frac{x}{e^x}=0$

$\alpha\beta + 2 > 0 \Rightarrow \boxed{\alpha\beta > -2}$

6. A straight line drawn from the point $P(1, 3, 2)$, parallel to the line $\frac{x-2}{1}=\frac{y-4}{2}=\frac{z-6}{1}$, intersects the plane $L_1 : x - y + 3z = 6$ at the point Q. Another straight line which passes through Q and is perpendicular to the plane L_1 intersects the plane $L_2 : 2x - y + z = -4$ at the point R.
Then which of the following statements is (are) TRUE?

(A) The length of the line segment PQ is $\sqrt{6}$

(B) The coordinates of R are $(1, 6, 3)$

(C) The centroid of the triangle PQR is $\left(\frac{4}{3},\frac{14}{3},\frac{5}{3}\right)$

(D) The perimeter of the triangle PQR is $\sqrt{2}+\sqrt{6}+\sqrt{11}$

Ans. **(A,C)**

Sol. line : $\frac{x-1}{1}=\frac{y-3}{2}=\frac{z-2}{1}$

$(x, y, z) = (\lambda + 1, 2\lambda + 3, \lambda + 2)$

Put in $L_1 : x - y + 3z = 6$

$(\lambda + 1) - (2\lambda + 3) + 3(\lambda + 2) = 6$

$2\lambda = 2 \Rightarrow \lambda = 1$

$\boxed{Q=(2,5,3)}$

line : $\frac{x-2}{1}=\frac{y-5}{-1}=\frac{z-3}{3}$

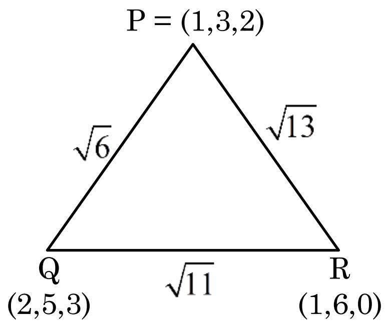
$(x, y, z) = (t + 2, 5 - t, 3t + 3)$

Put in $L_2 : 2x - y + z = -4$

$2(t + 4) - (5 - t) + (3t + 3) = -4$

$6t = -6 \Rightarrow t = -1$

$\boxed{R=(1,6,0)}$



Perimeter = $\sqrt{6}+\sqrt{13}+\sqrt{11}$

Centroid = $\left(\frac{4}{3},\frac{14}{3},\frac{5}{3}\right)$

7. Let A_1, B_1, C_1 be three points in the xy-plane. Suppose that the lines A_1C_1 and B_1C_1 are tangents to the curve $y^2 = 8x$ at A_1 and B_1, respectively. If $O = (0, 0)$ and $C_1 = (-4, 0)$, then which of the following statements is (are) TRUE?

(A) The length of the line segment OA_1 is $4\sqrt{3}$

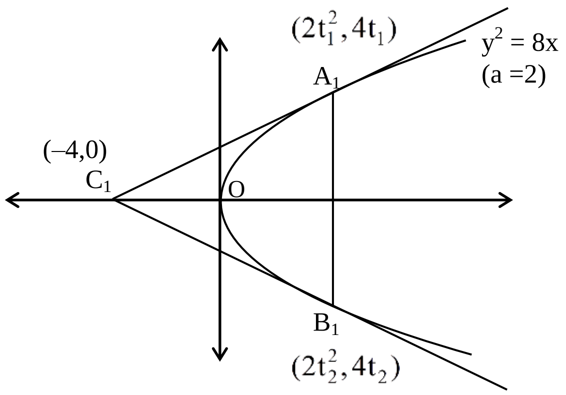
(B) The length of the line segment A_1B_1 is 16

(C) The orthocenter of the triangle $A_1B_1C_1$ is (0, 0)

(D) The orthocenter of the triangle $A_1B_1C_1$ is (1, 0)

Ans. **(A,C)**

Sol.



Equation of tangent at $(2t^2, 4t)$ is

$ty = x + 2t^2$

$\because$ It is passing through (–4, 0)

$\therefore\ 0 = -4 + 2t^2 \Rightarrow t = \pm\sqrt{2}$

$$\therefore \left.\begin{aligned} A_1 &= (4, 4\sqrt{2}) \\ B_1 &= (4, -4\sqrt{2}) \end{aligned}\right\} \quad \begin{aligned} OA_1 &= \sqrt{48} = 4\sqrt{3} \\ A_1B_1 &= 8\sqrt{2} \end{aligned}$$

Equation of altitude of $\Delta A_1B_1C_1$ drawn from A_1 is

$y - 4\sqrt{2} = \sqrt{2}(x - 4)$

$\Rightarrow \sqrt{2}x - y = 0 \qquad \ldots(1)$

Equation of altitude of $\Delta A_1B_1C_1$ drawn from C_1 is

$x = 0 \qquad \ldots(2)$

Solving (1) and (2) $\Rightarrow$ orthocentre is (0, 0)

$\therefore$ correct options are (A), (C)

SECTION-3 : (Maximum Marks : 24)

- This section contains **SIX (06)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER.**
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If **ONLY** the correct integer is entered;

Zero Marks : 0 In all other cases.

8. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$, and $g: \mathbb{R} \rightarrow (0, \infty)$ be a function such that $g(x+y) = g(x)\,g(y)$ for all $x, y \in \mathbb{R}$. If $f\left(\frac{-3}{5}\right) = 12$ and $g\left(\frac{-1}{3}\right) = 2$, then the value of $\left(f\left(\frac{1}{4}\right) + g(-2) - 8\right)g(0)$ is ________ .

Ans. **(51)**

Sol. $f(x+y) = f(x) + f(y)$ ...(1)

$\Rightarrow$ $f(nx) = nf(x)\ \forall\ n \in N$ ...(2)

Now put $y = -x$ in eq.(1)

$f(x) + f(-x) = f(0)$ $\{f(0) = 0\}$

$\Rightarrow$ $f(-x) = -f(x)$

$\Rightarrow$ f is odd function

from eq. (2)

$f(-nx) = nf(-x)$

$\Rightarrow$ $f(-nx) = -nf(x)$

$\Rightarrow$ $f(mx) = mf(x)\ \forall\ m \in Z^-$ ...(3)

from eq. (2) and eq. (3)

$f(nx) = nf(x)\ \forall\ n \in Z$ ...(4)

Now put $x = \frac{p}{q}$ where $p, q \in Z, q \neq 0$

$$f\left(\frac{np}{q}\right) = nf\left(\frac{p}{q}\right) \forall n \in Z$$

put $n = q$

$$f(p) = qf\left(\frac{p}{q}\right)$$

$\Rightarrow$ $pf(1) = qf\left(\frac{p}{q}\right)$ {from eq.(4)}

Let f(1) = a

then $pa = qf\left(\frac{p}{q}\right)$

$$f\left(\frac{p}{q}\right) = \frac{ap}{q}$$

$\Rightarrow$ $f(x) = ax \ \forall \ x \in \mathbb{Q}$

Now, $f\left(\frac{-3}{5}\right) = a\left(\frac{-3}{5}\right) = 12 \ \Rightarrow \ \boxed{a = -20}$

$\Rightarrow$ $f(x) = -20x \ \forall \ x \in \mathbb{Q}$ ...(5)

From the given functional equation it is not possible to find a unique function for irrational values of 'x', there are infinitely many such functions satisfying given functional equation for irrational values of x, but in this problem we finally need the function at rational values of 'x' only. So, for rational values of x we are getting a unique function mentioned in (5).

Now, g(x + y) = g(x).g(y)

$\Rightarrow$ $\ell n(g(x + y) = \ell n(g(x)) + \ell n(g(y))$

Let $\ell n(g(x)) = h(x)$

$\Rightarrow$ $h(x + y) = h(x) + h(y)$

$\Rightarrow$ $h(x) = kx \ \forall \ x \in \mathbb{Q}$

$\Rightarrow$ $g(x) = e^{kx} \ \forall \ x \in \mathbb{Q}$ ...(6)

and $g\left(\frac{-1}{3}\right) = e^{-\frac{K}{3}} = 2$ $\Rightarrow$ $K = -3\ell n2$

$\Rightarrow$ $K = \ell n\left(\frac{1}{8}\right)$

$\Rightarrow$ $g(x) = e^{\ell n\left(\frac{1}{8}\right).x} = \left(\frac{1}{8}\right)^{x} = 2^{-3x} \ \forall \ x \in \mathbb{Q}$

Now, $f\left(\frac{1}{4}\right) = -5, \ g(-2) = 2^{6} = 64$

g(0) = 1

So $\left(f\left(\frac{1}{4}\right) + g(-2) - (8).g(0)\right)$

= (–5 + 64 – 8) (1) = 51

9. A bag contains N balls out of which 3 balls are white, 6 balls are green, and the remaining balls are blue. Assume that the balls are identical otherwise. Three balls are drawn randomly one after the other without replacement. For $i = 1, 2, 3$, let W_i, G_i, and B_i denote the events that the ball drawn in the i^{th} draw is a white ball, green ball, and blue ball, respectively. If the probability $P(W_1 \cap G_2 \cap B_3) = \frac{2}{5N}$ and the conditional probability $P(B_3 | W_1 \cap G_2) = \frac{2}{9}$, then N equals ________ .

Ans. **(11)**

Sol.

3 White
6 Green
(N – 9) Blue

Given $P(W_1 \cap G_2 \cap B_3) = \frac{2}{5N}$

and $P(B_3 | W_1 \cap G_2) = \frac{2}{9}$

$$\Rightarrow \frac{P(B_3 \cap W_1 \cap G_2)}{P(W_1 \cap G_2)} = \frac{2}{9}$$

$$\Rightarrow \frac{2}{5N} \times \frac{N \times (N-1)}{3 \times 6} = \frac{2}{9}$$

$\Rightarrow N = 11$

10. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \frac{\sin x}{e^{\pi x}} \frac{(x^{2023} + 2024x + 2025)}{(x^2 - x + 3)} + \frac{2}{e^{\pi x}} \frac{(x^{2023} + 2024x + 2025)}{(x^2 - x + 3)}.$$

Then the number of solutions of $f(x) = 0$ in $\mathbb{R}$ is ______.

Ans. **(1)**

Sol. $$f(x) = \frac{(x^{2023} + 2024x + 2025)}{e^{\pi x}(x^2 - x + 3)}(\sin x + 2)$$

$\because$ (sinx + 2) is never zero

$\therefore$ for $x^{2023} + 2024x + 2025 = 0$

let $\phi(x) = x^{2023} + 2024x + 2025$

$\phi'(x) = 2023x^{2022} + 2024 > 0 \ \forall \ x \in R$

$\therefore$ $\phi(x)$ is an Strictly Increasing function

$\therefore$ $\phi(x) = 0$ for exactly one value of x

$\therefore$ f(x) = 0 has one solution

11. Let $\vec{p} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{q} = \hat{i} - \hat{j} + \hat{k}$. If for some real numbers α, β and γ, we have

$$15\hat{i} + 10\hat{j} + 6\hat{k} = \alpha(2\vec{p} + \vec{q}) + \beta(\vec{p} - 2\vec{q}) + \gamma(\vec{p} \times \vec{q}),$$

then the value of γ is ________ .

Ans. **(2)**

Sol. $15\hat{i} + 10\hat{j} + 6\hat{k} = \alpha(2\vec{p} + \vec{q}) + \beta(\vec{p} - 2\vec{q}) + \gamma(\vec{p} \times \vec{q})$

taking dot with $(\vec{p} \times \vec{q})$

$$\begin{vmatrix} 15 & 10 & 6 \\ 2 & 1 & 3 \\ 1 & -1 & 1 \end{vmatrix} = 0 + 0 + \gamma(p^2q^2 - (\vec{p}.\vec{q})^2) \qquad \left[\because\ (\vec{p} \times \vec{q})^2 + (\vec{p}.\vec{q})^2 = p^2q^2\right]$$

$\Rightarrow\ 52 = 26\,\gamma$

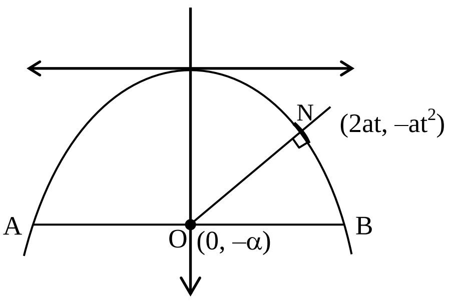
$\therefore\ \ \gamma = 2$

12. A normal with slope $\frac{1}{\sqrt{6}}$ is drawn from the point $(0, -\alpha)$ to the parabola $x^2 = -4ay$, where $a > 0$. Let L be the line passing through $(0, -\alpha)$ and parallel to the directrix of the parabola.

Suppose that L intersects the parabola at two points A and B. Let r denote the length of the latus rectum and s denote the square of the length of the line segment AB. If $r : s = 1 : 16$, then the value of $24a$ is ____.

Ans. **(12)**

Sol.



$$\frac{dy}{dx} = \frac{x}{-2a} \Rightarrow \left.\frac{dy}{dx}\right|_N = -t$$

Slope of normal = $\frac{1}{t} = \frac{1}{\sqrt{6}} \Rightarrow \boxed{t = \sqrt{6}}$

Now, $\frac{-at^2 + \alpha}{2at} = \frac{1}{t}$

$\Rightarrow\ -at^2 + \alpha = 2a$

$\Rightarrow\ -6a + \alpha = 2a\ \ \Rightarrow \alpha = 8a$

For A and B

$x^2 = -4a(-8a)$

$\Rightarrow x^2 = 32a^2 \quad \Rightarrow x = \pm 4\sqrt{2}a$

$\therefore\ A(-4\sqrt{2}a, -8a), B(4\sqrt{2}a, -8a)$

$\therefore\ AB^2 = (8\sqrt{2}a)^2 = 128\ a^2 = s$

$\therefore$ Length of LR = r = 4a

$\Rightarrow \frac{r}{s} = \frac{4a}{128a^2} = \frac{1}{16}$

$\therefore\ 32a = 16 \Rightarrow a = \frac{1}{2}$

$\therefore\ 24a = 12$ Ans.

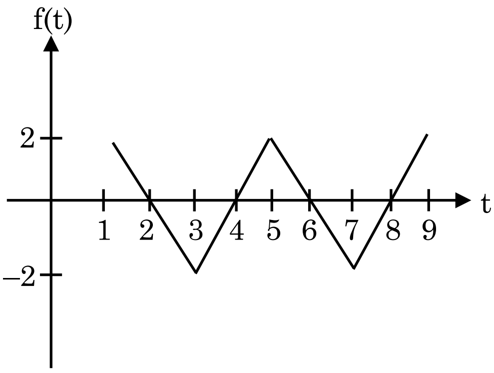
13. Let the function $f : [1, \infty) \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} (-1)^{n+1}2, & \text{if } t = 2n-1,\ n \in \mathbb{N}, \\ \frac{(2n+1-t)}{2} f(2n-1) + \frac{(t-(2n-1))}{2} f(2n+1), & \text{if } 2n-1 < t < 2n+1,\ n \in \mathbb{N} \end{cases}$$

Define $g(x) = \int_1^x f(t)dt$, $x \in (1, \infty)$. Let α denote the number of solutions of the equation $g(x) = 0$ in the interval (1, 8] and $\beta = \lim_{x \to 1^+} \frac{g(x)}{x-1}$. Then the value of $\alpha + \beta$ is equal to ______.

Ans. **(5)**

Sol.
$$f(t) = \begin{cases} 2 & ; \quad t = 1 \\ 4-2t & ; \quad 1 < t < 3 \\ -2 & ; \quad t = 3 \\ -8-2t & ; \quad 3 < t < 5 \\ 2 & ; \quad t = 5 \\ 12-2t & ; \quad 5 < t < 7 \\ -2 & ; \quad t = 7 \\ -16+2t & ; \quad 7 < t < 9 \end{cases}$$



$g(x) = \int_1^x f(t)dt \ ; \ g'(x) = f(x)$

for $x \in (1,8]$

$g(x) = 0 \Rightarrow x = 3,5,7 \ \therefore\ \alpha = 3$

$\beta = \lim_{x \to 1^+} \frac{g(x)}{x-1}$

Apply L'pital

$= \frac{g'(1^+)}{1} = f(1^+)$

$\beta = 2$

$\therefore\ \alpha + \beta = 5$

SECTION-4 : (Maximum Marks : 12)

- This section contains **TWO (02)** paragraphs.
- Based on each paragraph, there are **TWO (02)** questions.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 If ONLY the correct numerical value is entered in the designated place;

Zero Marks : 0 In all other cases.

"PARAGRAPH I"

Let $S = \{1, 2, 3, 4, 5, 6\}$ and X be the set of all relations R from S to S that satisfy both the following properties.

i. R has exactly 6 elements.

ii. For each $(a, b) \in R$, we have $|a - b| \geq 2$.

Let $Y = \{R \in X :$ The range of R has exactly one element$\}$ and

$Z = \{R \in X : R$ is a function from S to $S.\}$

Let $n(A)$ denote the number of elements in a set A.

(There are two questions based on PARAGRAPH "I", the question given below is one of them)

14. If $n(X) = {}^{m}C_6$, then the value of m is __________.

Ans. **(20.00)**

Sol. $|a-b| \geq 2$ or $|b-a| = 2$

		Total
$a = 1$	$b = 3,4,5,6$	8
$a = 2$	$b = 4,5,6$	6
$a = 3$	$b = 5,6$	4
$a = 4$	$b = 6$	2

sum = 20

$$n(X) = {}^{20}C_6 = {}^{m}C_6$$

$m = 20$

"PARAGRAPH I"

Let $S = \{1, 2, 3, 4, 5, 6\}$ and X be the set of all relations R from S to S that satisfy both the following properties.

i. R has exactly 6 elements.

ii. For each $(a, b) \in R$, we have $|a - b| \geq 2$.

Let $Y = \{R \in X : \text{The range of R has exactly one element}\}$ and

$Z = \{R \in X : R \text{ is a function from } S \text{ to } S.\}$

Let $n(A)$ denote the number of elements in a set A.

(There are two questions based on PARAGRAPH "I", the question given below is one of them)

15. If the value of $n(Y) + n(Z)$ is k^2, then $|k|$ is ___________.

Ans. (36.00)

15.

Sol. given $|a - b| \geq 2$ so if

$a = 1 \quad b = 3,4,5,6 \quad \rightarrow \quad 4 \times 2 = 8$

$a = 1 \quad b = 4,5,6 \quad \rightarrow \quad 3 \times 2 = 6$

$a = 1 \quad b = 5,6 \quad \rightarrow \quad 2 \times 2 = 4$

$a = 1 \quad b = 6 \quad \rightarrow \quad 2 \times 1 = 2$

20

i.e. Total elements in X is ${}^{20}C_6$

Now for n(Y),

range of R has exactly one element i.e. second elements must be constant in R and since R must have 6 element so it is not possible to satisfy both condition so n(Y) = 0.

for n(z)

$1 \rightarrow 3,4,5,6$

$2 \rightarrow 4,5,6$

$3 \rightarrow 1,5,6$

$4 \rightarrow 1,2,6$

$5 \rightarrow 1,2,3$

$6 \rightarrow 1,2,3,4$

no. of relation that are function will be $= {}^4C_1 \times {}^3C_1 \times {}^3C_1 \times {}^3C_1 \times {}^3C_1 \times {}^4C_1$

$= (4 \times 3 \times 3)^2 = k^2$

i.e. k = 36

"PARAGRAPH II"

Let $f : \left[0, \frac{\pi}{2}\right] \to [0,1]$ be the function defined by $f(x) = \sin^2 x$ and let $g : \left[0, \frac{\pi}{2}\right] \to [0,\infty)$ be the function defined by $g(x) = \sqrt{\frac{\pi x}{2} - x^2}$.

(There are two questions based on PARAGRAPH "II", the question given below is one of them)

16. The value of $2\int_0^{\frac{\pi}{2}} f(x)g(x)dx - \int_0^{\frac{\pi}{2}} g(x)dx$ is ____________.

Ans. **(0.00)**

Sol. $$I = 2\underbrace{\int_0^{\frac{\pi}{2}} \sin^2 x.\sqrt{\frac{\pi x}{2} - x^2}}_{I_1} - \int_0^{\frac{\pi}{2}} g(x)dx$$

Let $I_1 = \int_0^{\frac{\pi}{2}} \sin^2 x\sqrt{\left(\frac{\pi}{4}\right)^2 - \left(x - \frac{\pi}{4}\right)^2}$ (making perfect square)

apply kings

$$I_1 = \int_0^{\frac{\pi}{2}} \cos^2 x\sqrt{\left(\frac{\pi}{4}\right)^2 - \left(\frac{\pi}{2} - x\right)^2}$$

add both

$$2I_1 = \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{\pi}{4}\right)^2 - \left(x - \frac{\pi}{4}\right)^2}$$

i.e. $2I_1 = \int_0^{\frac{\pi}{2}} g(x)$

Now $I = 2I_1 - \int_0^{\frac{\pi}{2}} g(x) = 0$

"PARAGRAPH II"

Let $f : \left[0, \frac{\pi}{2}\right] \to [0,1]$ be the function defined by $f(x) = \sin^2 x$ and let $g : \left[0, \frac{\pi}{2}\right] \to [0,\infty)$ be the function defined by $g(x) = \sqrt{\frac{\pi x}{2} - x^2}$.

(There are two questions based on PARAGRAPH "II", the question given below is one of them)

17. The value of $\frac{16}{\pi^3}\int_0^{\frac{\pi}{2}} f(x)g(x)dx$ is _________.

Ans. **(0.25)**

Sol. Now $I_1 = \int_0^{\frac{\pi}{2}} f(x).g(x)dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} g(x)dx$

i.e. $\frac{1}{2}\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{\pi}{4}\right)^2 - \left(x - \frac{\pi}{4}\right)^2}\, dx$

Using $\int \sqrt{a^2 - x^2} = \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right) + C$

$$\Rightarrow \frac{1}{2}\left[\frac{\left(x - \frac{\pi}{4}\right)}{2}\sqrt{\frac{\pi x}{2} - x^2} + \frac{\frac{\pi^2}{16}}{2}\sin^{-1}\left(\frac{x - \frac{\pi}{4}}{\frac{\pi}{4}}\right)\right]_0^{\pi/2}$$

$$\Rightarrow \frac{1}{2}\left[\left(0 + \frac{\pi^3}{64}\right) - \left(0 + \left(\frac{-\pi^3}{64}\right)\right)\right]$$

$$\Rightarrow \frac{1}{2} \times \frac{\pi^3}{32}$$

Now $\frac{16}{\pi^3} \times \frac{\pi^3}{64} = \frac{1}{4} = 0.25$